1.7 – Diagonal, Triangular, and Symmetric Matrices

Definition: A square matrix in which all the entries off the main diagonal are zero is called a **diagonal matrix**.

#**3** Find the product by inspection.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}$$

#**8** Find A^2, A^{-2}, and A^k (where k is any integer) by inspection.

$$A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 36 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} \qquad A^{-2} = \underbrace{(A^{-1})^2}_{(A^2)^{-1}} = \begin{bmatrix} \frac{1}{36} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{25} \end{bmatrix}$$

$$A^k = \begin{bmatrix} (-6)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 5^k \end{bmatrix} \qquad A^{-k} = \begin{bmatrix} \frac{1}{(-6)^k} & 0 & 0 \\ 0 & \frac{1}{3^k} & 0 \\ 0 & 0 & \frac{1}{5^k} \end{bmatrix}$$

Definitions: A square matrix in which all entries below the main diagonal are zero is an **upper triangular** matrix. A square matrix in which all entries above the main diagonal are zero is a **lower triangular** matrix. If a matrix is either upper triangular or lower triangular (or both), it is said to be **triangular**.

diagonal

Consider $[a_{ij}]$, a square matrix.

$$A = \begin{bmatrix} d_1 & & i<j \\ & d_2 & \\ i>j & \ddots & d_n \end{bmatrix}$$

any nonzero entries are here in an upper triangular matrix

any nonzero entries are here in a lower triangular matrix

Theorem 1.7.1 Properties of Triangular Matrices

a) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

b) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

d) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Pf a) (lower triangular)

Let $A=[a_{ij}]$ be a lower triangular matrix. Then $a_{ij}=0$ if $i<j$, that is if the row number is less than the column number. But $A^T=[a_{ij}]^T=[a_{ji}]$, and $a_{ji}=0$ for $j>i$, that is if the row number is greater than the column number. A^T is then upper triangular.

#19 Determine by inspection whether the matrix is invertible.

$$\begin{bmatrix} 0 & 6 & -1 \\ 0 & 7 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

No.

#23 Find the diagonal entries of AB by inspection.

$$A = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

$(AB)_{11} = -3$ $\quad$ $(AB)_{22} = 5$ $\quad$ $(AB)_{33} = -6$

#46 Prove: If the matrices A and B are both upper triangular or both lower triangular, then the diagonal entries of both AB and BA are the products of the diagonal entries of A and B.

pf: (upper triangular, AB)

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ upper triangular matrices. Then $a_{ij} = 0$ if $i > j$ and $b_{ij} = 0$ if $i > j$.

$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. If $i > k$ (or $k < i$) $a_{ik} = 0$ and if $k > j$ $(j < k)$, $b_{kj} = 0$.

(Side note: $i > k > j \Rightarrow i > j \Rightarrow a_{ik} b_{kj} = 0$

so AB is upper triangular)

$(AB)_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$. But if $i \neq k$,

$a_{ik} b_{ki} = 0 \Rightarrow (AB)_{ii} = a_{ii} b_{ii}$

Definition: A square matrix A is said to be **symmetric** if $A = A^T$.

Theorem 1.7.2 Algebraic Properties of Symmetric Matrices
If A and B are symmetric matrices with the same size, and if k is any scalar, then:
a) A^T is symmetric.
b) $A + B$ and $A - B$ are symmetric.
c) kA is symmetric.

Theorem 1.7.3 The product of two symmetric matrices is symmetric if and only if the matrices commute. $(AB)^T = B^T A^T = BA \boxed{=} AB$

Theorem 1.7.4 If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Theorem 1.7.5 If A is an invertible matrix, then $\underline{AA^T}$ and $A^T A$ are also invertible.

product of 2 invertible matrices.

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 4 \\ 5 & 4 & 7 \end{bmatrix}$$ is symmetric